agreement with this data. In Ref. 1, however, the discrepancy is empirically corrected by inserting a square root of the Prandtl number in R. If this dependence is correct, it can be verified for a monotonic gas (which is free of rotational and vibrational relaxation effects) by using, for instance, a mixture of He and Ar. (Reference 1 shows data for both He and Ar but, unfortunately, not for a binary mixture.) Table 2 shows the Prandtl number for such a mixture computed by formulas contained in Ref. 6. The change in Prandtl number with mole fraction appears to be adequate for the proposed verification.

Acknowledgment

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References

[1]Tang, S. P. and Fenn, J. B., "Exerimental Determination of the Discharge Coefficients for Critical Flow Through an Axisymmetric Nozzle," *AIAA Journal*, Vol. 16, Jan. 1978, pp. 41-46.

[2]Tang, S., "Discharge Coefficients for Critical Flow Nozzles and Their Dependence on Reynolds Numbers," Ph.D. Thesis, Princeton Univ., 1969.

[3]Coles, D., "The Laminar Boundary Layer Near a Sonic Throat," *1957 Proceedings of the Heat Transfer and Fluid Mechanics Inst.*, Stanford University Press, Stanford, Calif., 1957, pp. 119-137.

[4]Emanuel, G., "Discharge Coefficient of a Chemical Laser Nozzle," *AIAA Journal*, Vol. 15, Jan. 1976, pp. 120-122.

[5]Kuluva, N. M. and Hosack, G. A., "Supersonic Nozzle Discharge Coefficients at Low Reynolds Numbers," *AIAA Journal*, Vol. 9, Sept. 1971, pp. 1876-1879.

[6]Holmes, J. T. and Baerns, M. G., "Predicting Physical Properties of Gas Mixtures," *Chemical Engineering*, Vol. 72, May 1965, pp. 103-108.

Reply by Authors to G. Emanuel

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WE appreciate Emanuel's favorable remarks but we are puzzled at his attempt to compare directly the result of Coles for a two-dimensional planar nozzle with that of Tang for the axisymmetric case. Surely it would be even more remarkable if there were no discrepancy between the A values of his Eqs, (4) and (5). Nor should it be surprising that Eq. (5), which relates to the axisymmetric nozzle, agrees better with our experimental data than does Eq. (4), which relates to the two-dimensional planar nozzle. Our data were obtained with axisymmetric nozzles.

With respect to the derivation of Kuluva and Hosack, we note that the pressure-gradient parameter β was not incorporated in their analysis. Thus, they did not assume any particular value. Our choice of the $\beta=\infty$ curve for comparison with the data stems from the fact that only for this case is a solution possible in closed form. Even more important, as pointed out in the paper,[1] the actual nozzle contours relate much more closely to $\beta=\infty$ than to other values. As we also pointed out, the theoretical C_D for $\beta=1$ is in much better agreement with the data than the C_D for $\beta=\infty$. Indeed, the agreement is even better than with the C_D values of Kuluva and Hosack, which, as Emanuel points out, do match the data better than our values of C_D for $\beta=\infty$. The point is that sheer consistency with experiment is not in itself a sufficient criterion for evaluating a theory. One can always force a better fit by empirical means of the kind used by Kuluva and Hosack in improving the fit to some of their data, or as we could in our case by arbitrarily picking the β value which gives the best fit. We felt that an empirical adjustment based on a modified Prandtl number was more defensible than choosing a β which was neither physically realistic nor analytically convenient.

In this connection, we find Emanuel's suggestion that measurements be made with mixtures of helium and argon in order to probe the true effect of Prandtl number interesting and constructive.

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References

[1]Tang, S.P. and Fenn, J.B., "Experimental Determination of the Discharge Coefficients for Critical Flow Through an Axisymmetric Nozzle," *AIAA Journal*, Vol. 16, Jan. 1978, pp. 41 – 46.

Comment on "Unconstrained Variational Statements for Initial and Boundary-Value Problems"

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THE first observation is a question of semantics directed to Simkins (and others—he is probably in good company) concerning the use of the word "variational." If a functional exists for which the vanishing of the first variation leads to a physical law, then there exists a variational principle. If a functional does not exist, then it is meaningless to speak of a variational statement. In reality, the formulation presented in Ref. 1 is nothing more than the method of weighted residuals with weighting functions expressed in terms of the trial functions of the approximation—that is, a generalized Galerkin formulation. For this procedure, if the approximation function is denoted by $\bar{y}(t)=\Sigma a_n\phi_n(t)$, then the weighting function is selected to be $\psi(t)=\Sigma b_n\phi_n(t)$. Next, if b_n is denoted by δa_n, it is possible to write $\psi(t)=\delta\bar{y}(t)$; and the appearance of the symbol "δ" would appear to be the only justification for the term "variational statement." It is the contention of this writer that such a term is confusing, nonprecise, and really unnecessary. When there is no functional, and therefore nothing to vary, then be precise and identify the formulation as one of the methods of weighted residuals.

Much of Ref. 1 is involved with identification of the Lagrange multipliers through integration by parts. When doing this, there is an implication (certainly there is no explicit statement to the contrary) that the Lagrange multipliers so obtained are the only possible values. In other words, there appears to be an implication that it is necessary to eliminate the so-called redundant boundary terms through suitable definitions of the $\delta\lambda$. However, after, the procedure is identified as a method of weighted residuals, then it becomes

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immediately obvious that there need be no relationship between boundary and domain weighting functions. Furthermore, as shown in Ref. 2, there need be no relationship between weighting functions and the trial functions which make up the approximate solution. These observations can be demonstrated with the following simple example of a particle moving with constant acceleration.

$$\text{for } t>0: \quad \ddot{y}=c \tag{1}$$

$$\text{for } t=0: \quad y=y_0, \quad \dot{y}=v_0 \tag{2}$$

A weighted residuals formulation begins with the equation

$$\int_0^t (\ddot{y}-c)\psi \mathrm{d}t + [y(0)-y_0]\alpha + [\dot{y}(0)-v_0]\beta = 0 \tag{3}$$

Assume

$$y(t)=a_0+a_1t+a_2t^2+a_3t^3+a_4t^4 \tag{4}$$

$$\psi(t)=b_1+b_2\sin t+b_3\cos t \tag{5}$$

and require that Eq. (3) be satisfied for arbitrary values of b_1, b_2, b_3, α, and β. The results are

$$a_0=y_0, \quad a_1=v_0, \quad a_2=\tfrac{1}{2}c, \quad a_3=0, \quad a_4=0$$

which gives

$$y(t)=y_0+v_0t+\tfrac{1}{2}ct^2 \tag{6}$$

Equation (6) is the correct answer, valid for all times t, as determined from a weighted residuals formulation in which the boundary weighting functions α and β are independent of the domain weighting function ψ and the ψ function does not have the same form as the trial function.

Simkins' Eq. (4), with obvious changes in notation, can also be used to solve the simple problem of Eqs. (1) and (2) above, with the trial solution of Eq. (4) above. It makes absolutely no difference in the final result whether one sets

$$\delta\lambda_1=\pm\delta y'(0) \text{ and } \delta\lambda_2=\pm\delta y(0) \text{ or } \delta\lambda_1=\pm\delta y(0)$$

$$\text{and } \delta\lambda_2=\pm\delta y'(0)$$

Each of the eight possibilities gives the exact results of Eq. (6) above.

This Comment is not suggesting that there is normally any advantage associated with such generality in weighting functions. The point is that the choices of Ref. 1 are not unique. Of course there is nothing wrong with associating boundary and domain weighting functions. Sometimes this makes it possible to achieve a formulation which reduces to a true variational formulation which might exist for a similar problem, and this might be desirable. Perhaps the search for the most accurate approximation according to some measure might provide guidance for choosing the Lagrange multipliers. However, as the example problem above shows, there is nothing really new or profound in the so-called unconstrained variational statements of Ref. 1. What we have are some very nice examples of the straightforward application of the method of weighted residuals applied to problems in a space-time domain.

Finally, a comment concerning the author's Eq. (12), which is stated to be valid for an elastic solid. More precisely, the indicated equality is valid only for a material with linear stress-strain laws experiencing sufficiently small deformations so that linear strain-displacement relations are satisfactory.

References

[1] Simkins, T. E., "Unconstrained Variational Statements for Initial and Boundary-Value Problems," *AIAA Journal*, Vol. 16, June 1978, pp. 559-563.

[2] Smith, C. V. and Smith, D. R., "Comment on 'Application of Hamilton's Law of Varying Action,' " *AIAA Journal*, Vol. 14, Feb. 1977, pp. 284-286.

Reply by Author to C. V. Smith Jr.

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MANY references can be cited to demonstrate that the existence of a functional is not required to state a physical problem in variational form. One such example is Hamilton's Law of Varying Action[1]:

$$\delta\int_{t_0}^{t_1}(T+W)\mathrm{d}t-\frac{\partial T}{\partial\dot{q}_i}\delta q_i\Big]_{t_0}^{t_1}=0$$

which only becomes the extremum principle of Hamilton if the second term vanishes. Even the principle of virtual work[2] is generally not an extremum principle. Still other examples may be found in the text of Leipholz.[3]

Lanczos (Ref. 2, p. 66), however, gives the clearest contradiction to Smith's understanding that only $\delta I=0$ represents a meaningful variational statement: "A variational problem with non-holonomic auxiliary conditions cannot be reduced to a form where the variation of a certain quantity is put equal to zero. However, the equations of motion are once more derivable, with the help of the [Lagrange] multiplier method, in a fashion analogous to the case of holonomic conditions."

Lanczos offers no explanation as to how the Lagrange multipliers are to be determined for general nonholonmic systems. The application of the Lagrange multiplier method to the nonholonomic initial conditions was, in fact, the subject of my paper.[4] The result presented is a variational formulation applicable to a large class of initial boundary-value problems. Upon performing the indicated variations, the appropriate equations of motion together with all boundary and initial conditions, are generated—indicating that the formulation represents a correct physical balance and may be useful as a basis for obtaining approximate solutions to such problems.

In reply to Smith's contention that it is really the method of weighted residuals (MWR) which has been utilized, one should realize that MWR requires that a choice be made as to how the residuals are to be weighted and what weighting functions are to be employed. Although there is no proof that best or even unique results will be obtained, many prefer that these choices be dictated by physical considerations. A variational statement representing a physical law or balance makes these choices *automatically*. This in itself is sufficient justification for distinguishing between variational methods (with or without a functional) and MWR. It is not surprising that different weighting functions, whether they have been introduced arbitrarily or by variational methods, can be used to achieve approximate and occasionally even exact solutions. Smith's example, however, is not a good one, since his assumed polynomial approximation has the form of the exact solution to begin with.

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